S.-T. Yau College Student Mathematics Contests 2020

Analysis and Differential Equations

Individual-I

Please solve the following problems.

1. (a) Assume that f is a non-decreasing function in $[0,1]$. Show that f' is integrable and that $\int_0^1 f'(x)dx \le f(1) - f(0)$. (b) Let $f_n(x)$ be a sequence of non-decreasing functions in $[0,1]$, and $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [0,1]$. Show that $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$ almost everywhere.

2. Suppose that $\Omega \subseteq \mathbb{C}$ is a domain and the unit disk $\overline{\mathbb{D}} \subseteq \Omega$. Let $f : \Omega \to \mathbb{C}$ be holomorphic with $f(0) = 0$, and $|f(e^{i\theta})| \ge 3$ for any $\theta \in \mathbb{R}$.

If $\lambda_1, \cdots, \lambda_N$ are all zeroes (counting multiplicities) of $1 - f(z)$ in $\mathbb{D}$, then we have

$$|\lambda_1 \cdots \lambda_N| < \frac{1}{2}.$$

3. Let u satisfy

$$\sum_{i,j=1}^{n} a_{ij} u_{ij} - u \ge 0 \text{ in } \mathbb{R}^n$$

where $(a_{ij}) > 0$ is a bounded and positive definite matrix. Furthermore assume that

$$u(x) \le 2020 + |x|^{2020}.$$

Show that

$$u(x) \le 0 \text{ in } \mathbb{R}^n.$$